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1989 J. Phys. A: Math. Gen. 22 3423

(http://iopscience.iop.org/0305-4470/22/16/031)

J. Phys. A: Math. Gen. 22 (1989) 3423-3428. Printed in the UK

COMMENT

Permutation operators in Hilbert space gained via the IWOP technique—fermion case

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Received 15 March 1989

Abstract. For the case of fermions, permutation operators in Hilbert space are derived in terms of the fermion coherent state and the technique of integration within ordered products with the integration variables being Grassmann numbers. These operators are shown to be quantum maps imaged by a certain permutation transformation of the numbers. Some new Fermi operator identities are also obtained with the use of the same technique.

1. Introduction

In a previous work [1], by exploiting the newly developed technique of integration within ordered products (IWOP) [2] we derived the unitary operators for permutations of N bosons in Hilbert space. An interesting question arises naturally: if the particles under study are fermions, what are the permutation operators in Hilbert space for N fermions? The purpose of this comment is to solve this problem by introducing the fermion coherent state and using the IWOP technique where the integration will be performed over the so-called anticommuting c numbers (or linear elements of a Grassmann algebra). In § 2, we list some properties of normal products of Fermi operators associated with Grassmann numbers and introduce IWOP technique into the fermion coherent state theory. In § 3, we derive the transposition operator because any permutation is equivalent to a finite number of transpositions. A new Fermi operator identity is also deduced in this section. The approach we use in § 3 can be directly generalised to derive N-fermion permutation operators, as shown in § 4.

2. Properties of normal products of Fermi operators

The following properties of normal products of Fermi operators are useful in this work.

(i) Any two Fermi operators anticommute with each other within a normal product, which has been known for a long time.

(ii) A Grassmann number-Fermi operator pair (GFP), say $\alpha_1 a_1$, commutes with another GFP within a normal product, e.g.

$$:\alpha_1 a_1 \alpha_2 a_2: = :\alpha_2 a_2 \alpha_1 a_1:. \tag{2.1}$$

0305-4470/89/163423+06$02.50

(iii) A normal product of Fermi operators can be integrated with respect to Grassmann variables according to the following formula [3]:

$$\int \prod_1^N \mathrm{d}\bar{\alpha}_i\, \mathrm{d}\alpha_i \exp\left[-\sum_{i,j} \bar{\alpha}_i \Lambda_{ij} \alpha_j + \sum_i (\bar{\alpha}_i \eta_i + \bar{\eta}_i \alpha_i)\right]$$

$$= \det \Lambda \exp\left[\sum_{i,j} \bar{\eta}_i (\Lambda^{-1})_{ij} \eta_j\right] \tag{2.2}$$

where Λ, as pointed out in [4], is a complex-valued matrix.

Consider now a $2N$-dimensional fermionic phase space spanned by the operators a_i and $a_i^\dagger$ $(i = 1, 2, \ldots, N)$ satisfying

$$\{a_i, a_j^\dagger\} = \delta_{ij} \qquad a_i^2 = a_i^{\dagger 2} = 0 \qquad \{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0. \tag{2.3}$$

The fermion coherent state is defined as [5]

$$a_i|\alpha_i\rangle = |\alpha_i\rangle \alpha_i \qquad \langle \alpha_i | a_i^\dagger = \bar{\alpha}_i \langle \alpha_i |.$$

Consistency requires α_i to be Grassmann numbers which anticommute with a_i and $a_i^\dagger$. α_i and $\bar{\alpha}_i$ obey

$$\alpha_i^2 = \bar{\alpha}_i^2 = 0 \qquad \alpha_i \bar{\alpha}_i + \bar{\alpha}_i \alpha_i = 0 \qquad \int \mathrm{d}\alpha_i\, \alpha_i = 1 \qquad \int \mathrm{d}\bar{\alpha}_i\, \bar{\alpha}_i = 1$$

$$\int \mathrm{d}\alpha_i = \int \mathrm{d}\bar{\alpha}_i = 0. \tag{2.4}$$

$|\alpha_i\rangle$ may then be represented by

$$|\alpha_i\rangle = \exp[-\tfrac{1}{2}\bar{\alpha}_i \alpha_i + a_i^\dagger \alpha_i]|0\rangle_i \tag{2.5}$$

where $|0\rangle_i$ obeys $a_i|0\rangle_i = 0$, and

$$|0\rangle_{ii}\langle 0| = :\mathrm{e}^{-a_i^\dagger a_i}:. \tag{2.6}$$

The orthogonality and completeness relations for $|\alpha_i\rangle$ are given by

$$\langle \alpha_i' | \alpha_i \rangle = \exp[-\tfrac{1}{2}(\bar{\alpha}_i \alpha_i + \bar{\alpha}_i' \alpha_i') + \bar{\alpha}_i' \alpha_i] \tag{2.7}$$

$$\int \mathrm{d}\bar{\alpha}_i\, \mathrm{d}\alpha_i |\alpha_i\rangle\langle\alpha_i| = \int \mathrm{d}\bar{\alpha}_i\, \mathrm{d}\alpha_i \exp(-\bar{\alpha}_i \alpha_i)(|0\rangle_i + |1\rangle_i \alpha_i)({}_i\langle 0| + \bar{\alpha}_i \langle 1|) = 1. \tag{2.8}$$

Using (2.5), (2.6) and (2.2), as well as the IWOP technique we can put (2.8) into the following normally ordered form:

$$\int \mathrm{d}\bar{\alpha}_i\, \mathrm{d}\alpha_i |\alpha_i\rangle\langle\alpha_i|$$

$$= \int \mathrm{d}\bar{\alpha}_i\, \mathrm{d}\alpha_i :\exp[-\bar{\alpha}_i \alpha_i + a_i^\dagger \alpha_i + \bar{\alpha}_i a_i - a_i^\dagger a_i]: = :\exp[a_i^\dagger a_i - a_i^\dagger a_i]: = 1. \tag{2.9}$$

Some applications of fermion coherent states are shown in [6].

3. Transposition operator and a new Fermi operator identity

Let the two-mode fermion coherent state be

$$|\alpha_1 \alpha_2\rangle = \exp[-\tfrac{1}{2}(\bar{\alpha}_1 \alpha_1 + \bar{\alpha}_2 \alpha_2) + a_1^\dagger \alpha_1 + a_2^\dagger \alpha_2]|00\rangle \qquad |00\rangle \equiv |0\rangle_1 |0\rangle_2 \tag{3.1}$$

and $|\alpha_2\alpha_1\rangle$ be

$$|\alpha_2\alpha_1\rangle = \exp[-\tfrac{1}{2}(\bar{\alpha}_1\alpha_1+\bar{\alpha}_2\alpha_2)+a_1^\dagger\alpha_2+a_2^\dagger\alpha_1]|00\rangle. \tag{3.2}$$

Using (3.1), (3.2) and the IWOP technique we perform the following integration:

$$\begin{aligned}P_{12} &\equiv \int d\bar{\alpha}_1\, d\alpha_1 \int d\bar{\alpha}_2\, d\alpha_2 |\alpha_2\alpha_1\rangle\langle\alpha_1\alpha_2| \\ &= \int d\bar{\alpha}_1\, d\alpha_1 \int d\bar{\alpha}_2\, d\alpha_2 \colon \exp[-\bar{\alpha}_1\alpha_1-\bar{\alpha}_2\alpha_2+a_1^\dagger\alpha_2+a_2^\dagger\alpha_1 \\ &\quad +\bar{\alpha}_1 a_1+\bar{\alpha}_2 a_2 - a_1^\dagger a_1 - a_2^\dagger a_2]\colon \\ &= \colon\exp[a_2^\dagger a_1 + a_1^\dagger a_2 - a_1^\dagger a_1 - a_2^\dagger a_2]\colon . \end{aligned} \tag{3.3}$$

In order to remove the symbol : : in (3.3), we prove the following new Fermi operator identity:

$$\exp\left[\sum_{ij} a_i^\dagger \Lambda_{ij} a_j\right] = \colon\exp\left[\sum_{ij} a_i^\dagger (e^\Lambda - \mathbb{1})_{ij} a_j\right]\colon . \tag{3.4}$$

Actually, with the aid of the following two operator identities:

$$e^A B\, e^{-A} = B + [A, B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \dots \tag{3.5}$$

$$[AB, C] = A\{C, B\} - \{C, A\}B \tag{3.6}$$

we can have

$$\exp\left[\sum_{ij} a_i^\dagger \Lambda_{ij} a_j\right] a_l^\dagger \exp\left[-\sum_{ij} a_i^\dagger \Lambda_{ij} a_j\right] = \sum_i a_i^\dagger (e^\Lambda)_{il} \tag{3.7}$$

where (2.3) is used. Further, in terms of (2.8) and the IWOP technique we can expand $\exp(\Sigma_{ij}\, a_i^\dagger \Lambda_{ij} a_j)$ as

$$\begin{aligned}\exp&\left[\sum_{ij} a_i^\dagger \Lambda_{ij} a_j\right] \\ &= \int \prod_1^N d\bar{\alpha}_i\, d\alpha_i \exp\left[\sum_{ij} a_i^\dagger \Lambda_{ij} a_j\right] \exp\left[\sum_i a_i^\dagger \alpha_i\right] \\ &\quad \times \exp\left[-\sum_{ij} a_i^\dagger \Lambda_{ij} a_j\right] |00\dots 0\rangle\langle\alpha_1\alpha_2\dots\alpha_N| \exp\left[-\frac{1}{2}\sum_i \bar{\alpha}_i\alpha_i\right] \\ &= \int \prod_1^N d\bar{\alpha}_i\, d\alpha_i \colon\exp\left[\sum_i\left(-\bar{\alpha}_i\alpha_i + \sum_j a_j^\dagger (e^\Lambda)_{ji}\alpha_i + \bar{\alpha}_i a_i - a_i^\dagger a_i\right)\right]\colon \\ &= \colon\exp\left[\sum_{ij} a_i^\dagger (e^\Lambda - \mathbb{1})_{ij} a_j\right]\colon \qquad \mathbb{1}\text{: } N\times N \text{ unit matrix.}\end{aligned} \tag{3.8}$$

Thus (3.4) is verified. As a result of (3.4), (3.3) can be reduced to

$$P_{12} = \exp\left[\frac{i\pi}{2}(a_1^\dagger - a_2^\dagger)(a_1 - a_2)\right]. \tag{3.9}$$

As a consequence of (2.3), it is easily seen that

$$\left\{\frac{a_1-a_2}{\sqrt{2}},\frac{a_1^\dagger-a_2^\dagger}{\sqrt{2}}\right\}=1 \qquad \left[\frac{1}{\sqrt{2}}(a_1-a_2)\right]^2=0$$
$$[\tfrac{1}{2}(a_1^\dagger-a_2^\dagger)(a_1-a_2)]^2=\tfrac{1}{2}(a_1^\dagger-a_2^\dagger)(a_1-a_2). \tag{3.10}$$

It then follows that

$$P_{12}=1+\left[\mathrm{i}\pi+\frac{(\mathrm{i}\pi)^2}{2!}+\frac{(\mathrm{i}\pi)^3}{3!}+\ldots\right]\frac{(a_1^\dagger-a_2^\dagger)}{\sqrt{2}}\frac{(a_1-a_2)}{\sqrt{2}}$$
$$=1-(a_1^\dagger-a_2^\dagger)(a_1-a_2)=P_{12}^\dagger=P_{12}^{-1} \tag{3.11}$$

$$P_{12}a_1P_{12}^{-1}=a_2 \qquad P_{12}a_2P_{12}^{-1}=a_1 \tag{3.12}$$

which confirms that (3.3) is indeed the transposition operator. Equation (3.3) also shows that P_{12} is the quantum map imaged by mutually interchanging $\alpha_1\leftrightarrow\alpha_2$ in Grassmann number space. In this way, we obtain the explicit form of P_{12} shown in (3.9) and (3.11), which is manifestly unitary.

4. *N*-fermion permutation operators

By analogy with the N-boson case shown in [1], we introduce the N-body permutation matrix

$$(uv\ldots w)=\begin{pmatrix}\delta_{u1} & \delta_{u2} & \ldots & \delta_{uN}\\ \delta_{v1} & \delta_{v2} & \ldots & \delta_{vN}\\ \vdots & & & \vdots\\ \delta_{w1} & \delta_{w2} & \ldots & \delta_{wN}\end{pmatrix} \tag{4.1}$$

where $u, v, \ldots, w$ is an arbitrary permutation of the numbers $1, 2, \ldots, N$. There exist $N!$ permutation matrices which constitute, in the sense of matrix products, a group. By rewriting the N-mode fermion coherent state as

$$\left|\begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_N\end{pmatrix}\right\rangle\equiv\prod_1^N|\alpha_i\rangle=|\alpha_1\rangle|\alpha_2\rangle\ldots|\alpha_N\rangle\equiv|\alpha_1\alpha_2\ldots\alpha_N\rangle \tag{4.2}$$

we can construct the permutation operator $P_{uv\ldots w}$ by

$$P_{uv\ldots w}=\int\prod_1^N \mathrm{d}\bar{\alpha}_i\,\mathrm{d}\alpha_i\left|(uv\ldots w)\begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_N\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_N\end{pmatrix}\right|$$
$$=\int\prod_1^N \mathrm{d}\bar{\alpha}_i\,\mathrm{d}\alpha_i\left|\begin{pmatrix}\alpha_u\\ \alpha_v\\ \vdots\\ \alpha_w\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_N\end{pmatrix}\right|. \tag{4.3}$$

Again, using the IWOP technique, we obtain the normally ordered form of $P_{uv\ldots w}$:

$$P_{uv\ldots w} = :\exp\left\{(a_1^\dagger a_2^\dagger \ldots a_N^\dagger)[(uv\ldots w) - \mathbb{1}]\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}\right\}: \tag{4.4}$$

which, by virtue of (3.4), becomes

$$P_{uv\ldots w} = \exp\left\{(a_1^\dagger a_2^\dagger \ldots a_N^\dagger)\ln(uv\ldots w)\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}\right\}. \tag{4.5}$$

To prove the unitarity of $P_{uv\ldots w}$, by using (2.7) and (4.3), we consider

$$P_{uv\ldots w}P^\dagger_{uv\ldots w} = \int \prod_1^N \mathrm{d}\bar{\alpha}_i\, \mathrm{d}\alpha_i \int \prod_1^N \mathrm{d}\bar{\alpha}'_i\, \mathrm{d}\alpha'_i \left|\begin{pmatrix} \alpha_u \\ \alpha_v \\ \vdots \\ \alpha_w \end{pmatrix}\right\rangle$$

$$\times \left\langle\begin{pmatrix} \alpha'_u \\ \alpha'_v \\ \vdots \\ \alpha'_w \end{pmatrix}\right| \exp\left\{\sum_i [-\tfrac{1}{2}(\bar{\alpha}_i\alpha_i + \bar{\alpha}'_i\alpha'_i) + \bar{\alpha}_i\alpha'_i)\right\}. \tag{4.6}$$

Due to (2.5) and (2.4) we have

$$\exp(-\tfrac{1}{2}\bar{\alpha}_i\alpha_i)\int \mathrm{d}\bar{\alpha}'_i\, \mathrm{d}\alpha'_i\langle\alpha'_i|\exp[-\tfrac{1}{2}\bar{\alpha}'_i\alpha'_i + \bar{\alpha}_i\alpha'_i] = \langle\alpha_i|. \tag{4.7}$$

With the help of (4.7) and $|\det(uv\ldots w)| = 1$, (4.6) becomes

$$P_{uv\ldots w}P^\dagger_{uv\ldots w} = 1 \tag{4.8}$$

which indicates that $P_{uv\ldots w}$ is unitary. For $n = 3$, from (4.5) we have

$$P_{312} = \exp\left[(a_1^\dagger a_2^\dagger a_3^\dagger)\ln\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\right]$$

$$= \exp\left[(a_1^\dagger a_2^\dagger a_3^\dagger)\begin{pmatrix} \gamma & \alpha & -\alpha^* \\ -\alpha^* & \gamma & \alpha \\ \alpha & -\alpha^* & \gamma \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\right] \qquad \alpha = \frac{1-\sqrt{3}\mathrm{i}}{3\sqrt{3}}\pi \qquad \gamma = \frac{2\pi \mathrm{i}}{3} \tag{4.9}$$

which is manifestly unitary. Though we have $a_i^2 = 0$ for fermion, it is not so easy to make a neat expansion of the exponential operator in (4.9) as we did for (3.9).

Nevertheless, using (2.7) and (4.7) we can obtain

$$
\begin{aligned}
P_{213}P_{132} &= \int \prod_1^3 \mathrm{d}\bar{\alpha}_i\,\mathrm{d}\alpha_i \left|\begin{pmatrix}\alpha_2\\ \alpha_1\\ \alpha_3\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}\right| \int \prod_1^3 \mathrm{d}\bar{\alpha}'_i\,\mathrm{d}\alpha'_i \left|\begin{pmatrix}\alpha'_1\\ \alpha'_3\\ \alpha'_2\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}\alpha'_1\\ \alpha'_2\\ \alpha'_3\end{pmatrix}\right| \\
&= \int \prod_1^3 \mathrm{d}\bar{\alpha}_i\,\mathrm{d}\alpha_i \left|\begin{pmatrix}\alpha_2\\ \alpha_1\\ \alpha_3\end{pmatrix}\right\rangle \int \prod_1^3 \mathrm{d}\bar{\alpha}'_i\,\mathrm{d}\alpha'_i \\
&\quad \times \exp\left[-\frac{1}{2}\sum_{i=1}^3 (\bar{\alpha}_i\alpha_i + \bar{\alpha}'_i\alpha'_i) + \bar{\alpha}_1\alpha'_1 + \bar{\alpha}_2\alpha'_3 + \bar{\alpha}_3\alpha'_2\right]\left\langle\begin{pmatrix}\alpha'_1\\ \alpha'_2\\ \alpha'_3\end{pmatrix}\right| \\
&= \int \prod_1^3 \mathrm{d}\bar{\alpha}_i\,\mathrm{d}\alpha_i \left|\begin{pmatrix}\alpha_3\\ \alpha_1\\ \alpha_2\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}\right| = P_{312}
\end{aligned}
\tag{4.10}
$$

which means the breakdown of P_{312} into the product of P_{213} and P_{132}. Since both P_{312} and $P_{213}P_{132}$ now have their own explicit Fermi operator forms in Hilbert space, as a by-product, one can obtain some new operator identities regarding the relationship between the permutation operator and its corresponding transposition operators.

In summary, we have seen that the IWOP technique with the integral variables being Grassmann numbers enlarges the field of application of fermion coherent states; the unitary permutation operators for N-fermion permutation are obtained in a closely parallel manner to the boson case in [1] (where, instead of using the boson coherent state, we used the coordinate eigenstate to realise the IWOP technique). The formulation in [1] and in this comment provides us with a fresh view—all unitary permutation operators in quantum statistics, either for bosons or for fermions, are quantum maps imaged by certain permutation transformations in classical space. Moreover, the classical-to-quantum transition is manifestly apparent in the formulation. We will give some new applications of the IWOP technique for Grassmann numbers in the future.

Note added in proof. Because any two Fermi operators are anticommuting with each other within : :, it is convenient to obtain the explicit operator form of P_{312} by directly expanding (4.4); the result is

$$
\begin{aligned}
P_{312} = 1 &+ a_1^\dagger a_3 a_2 a_2^\dagger + a_2^\dagger a_1 a_3 a_3^\dagger + a_3^\dagger a_2 a_1 a_1^\dagger \\
&- a_1^\dagger a_1 a_2 a_2^\dagger - a_2^\dagger a_2 a_3 a_3^\dagger - a_3^\dagger a_3 a_1 a_1^\dagger - a_1^{a\dagger} a_1 a_2^\dagger a_3 - a_2^\dagger a_2 a_3 a_1 - a_3^\dagger a_3 a_1^\dagger a_2 .
\end{aligned}
$$

One can directly check that

$$
P_{312} a_1 P_{312}^\dagger = a_2 \qquad P_{321} a_2 P_{312}^\dagger = a_3 \qquad P_{312} a_3 P_{312}^\dagger = a_1 .
$$

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